Approximation of Systems of Volterra Integral Equations of the Second Kind Using the New Iterative Method

Terhemen ABOIYAR
Department of Mathematics/Statistics/Computer Science
University of Agriculture, Makurdi,
Nigeria
t_aboiyar@yahoo.co.uk

Hasssan IBRAHIM
Department of Mathematics
Federal University Lafia, Lafia,
Nigeria.
equationxyz4real@gmail.com

ABSTRACT

In this article, we solve linear and nonlinear system of Volterra integral equations using the New Iterative Method. This method provides a solution in a rapidly convergent series form. The convergence analysis of the method is also considered. Furthermore, the efficiency of the method will be demonstrated through numerical examples.

Keywords: New Iterative Method, System of Volterra Integral Equations, Lipschitz condition.

1. INTRODUCTION

Systems of Volterra integral equations, linear or nonlinear, are used in modelling various physical phenomena in science and engineering. Finding the exact solution of the system of Volterra integral equations by classical methods is sometimes too difficult, and is usually very useful to find a numerical estimation of the exact solution(Mohammad, 2012).Different techniques have been used for solving the systems of Volterra integral equations, such as Differential transform method(Biazar and Eslami, 2011), New Modification Adomian decomposition's method (NMADM) (Lie-jun, 2013),Toeplitz matrix method (Handi and Al-Hazmi, 2010), He's Homotopy perturbation method(Buazar and Ghazvini, 2009), Laplace Adomain Decomposition Method (LADM) (Handi, 2011).

In this paper, we will consider the following linear system of Volterra integral equations of the second kind:

$$\left.\begin{array}{l} y_1(x) = f_{1(x)} + \int_0^x \big(K_{11}(x,t)y_1(t) + K_{12}(x,t)y_2(t) + \cdots + K_{1n}(x,t)y_n(t)\big)\,dt, \\ y_2(x) = f_{2(x)} + \int_0^x \big(K_{21}(x,t)y_1(t) + K_{22}(x,t)y_2(t) + \cdots + K_{2n}(x,t)y_n(t)\big)\,dt, \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ y_n(x) = f_{n(x)} + \int_0^x \big(K_{n1}(x,t)y_1(t) + K_{n2}(x,t)y_2(t) + \cdots + K_{nn}(x,t)y_n(t)\big)\,dt \end{array}\right\} \quad (1)$$

where $y_1, y_2, \dots y_n$ are the unknown functions, $f_1, f_2, \dots f_n$ are the given functions and $K_{11}, K_{21}, \dots, K_{n1}$ are called the kernel or nuclei of integral equations.

Systems of nonlinear Volterra integral equations of the second kind are of the form:

$$\left.\begin{aligned}
y_1(x) &= f_1(x) + \int_0^x \Big(K_1\big(x,t,y_1(t),\ y_2(t),\dots,y_n(t)\big)\Big)\,dt,\\
y_2(x) &= f_2(x) + \int_0^x \Big(K_2\big(x,t,y_1(t),\ y_2(t),\dots,y_n(t)\big)\Big)\,dt,\\
&\ \ \vdots\\
y_n(x) &= f_n(x) + \int_0^x \Big(K_n\big(x,t,y_1(t),\ y_2(t),\dots,y_n(t)\big)\Big)\,dt
\end{aligned}\right\} \quad (2)$$

In this paper, we apply the new iterative method proposed by Daftardar-Gejji and Jafari (2006) for solving the above system, both linear and nonlinear.

2. The New Iterative Method

To describe the idea of the new iterative method (NIM), we consider the following general formulation by Dafatardar-Gejji and Jafari (2006). Consider the nonlinear functional equation:

$$y(x) = f(x) + N\big(y(x)\big) \quad (3)$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. We are looking for a solution $y(x)$ of (3) having the series form:

$$y(x) = \sum_{i=0}^{\infty} y_i \quad (4)$$

The nonlinear operator N can be decomposed as follows:

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \quad (5)$$

From Eq. (4) and (5), Eq. (3) is equivalent to

$$N\left(\sum_{i=0}^{\infty} y_i\right) = f(x) + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \quad (6)$$

we define the recurrence relation:

$$\left.\begin{aligned}
y_0 &= f(x)\\
y_1 &= N(y_0)\\
y_2 &= N(y_0 + y_1) - N(y_0)\\
&\ \ \vdots
\end{aligned}\right\} \quad (7)$$

.

$$y_{m+1} = N(y_0 + y_1 + \cdots + y_m) - N(y_0 + y_1 + \cdots + y_{m-1}),\ m = 1,2,\ldots$$

then

$$y_0 + y_1 + \cdots + y_{m+1} = N(y_0 + y_1 + \cdots + y_m), m = 1,2,\ldots, \tag{8}$$

and

$$y(x) = f(x) + \sum_{i=0}^{\infty} y_i \tag{9}$$

If N is a contraction, i.e.

$$\|N(x) - N(y)\| \le k\|x - y\|, 0 < k < 1,$$

then

$$\|y_{m+1}\| = \|N(y_0 + y_1 + \cdots + y_m) - N(y_0 + y_1 + \cdots + y_{m-1})\|$$

$$\le k\|y_m\| \le \cdots \le k^n\|y_0\|,\ m = 0,1,2,\ldots,$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to solution of Eq. (1) and Eq. (2)., which is unique, in view of Banach fixed point theorem(Jerri, 1999). The k-term approximate solution of Eq. (3) and (4) is given by $\sum_{i=0}^{k-1} y_i$.

3. New Iterative Method for the Systems of Volterra Integral Equations.

We wil first of all rewrite the system of linear and nonlinear Volterra integral equations in Eq. (1) and (2) in vector form. Eq.(1) can be written in vector form as:

$$\boldsymbol{y}(x) = \boldsymbol{f}(x) + \int_0^x \boldsymbol{K}(x,t)\boldsymbol{y}(t)dt \tag{10}$$

where

$$\boldsymbol{f}(x) = [f_1(x),\ f_2(x), \ldots, f_n(x)]^T,$$
$$\boldsymbol{y}(t) = [y_1(t),\ y_2(t), \ldots, y_n(t)]^T, \text{ and}$$
$$\boldsymbol{K}(x,t) = \left[K_{ij}(x,t)\right], i,j = 1,2,\ldots n$$

The kernels or nuclei $\left[\boldsymbol{K}_{ij}(x,t)\right], i,j = 1,2,\ldots n$ and the functions $f_i(x), i = 1,2,\ldots,n$ are given real-valued functions.

In a vector notation, Eq. (2) can be written as:

$$\boldsymbol{y}(x) = \boldsymbol{f}(x) + \int_0^x \boldsymbol{K}\big(x,t,\boldsymbol{y}(t)\big)dt \tag{11}$$

where

$$\boldsymbol{f}(x) = [f_1(x),\ f_2(x), \ldots, f_n(x)]^T,$$

$\boldsymbol{y}(t) = [y_1(t),\ y_2(t), \dots, y_n(t)]^T$, and
$\mathbf{K} = [K_1, K_2, \dots, K_n]$

The kernels or nuclei $\mathbf{K} = [K_1, K_2, \dots, K_n]$ and the functions $f_i(x), i = 1,2,\dots,n$ are given real-valued functions.

The New Iterative Algorithm is applied to (10) as follows:

$$\left.\begin{aligned}
&\boldsymbol{y_0}(x) = \boldsymbol{f}(x)\\
&\boldsymbol{y_1}(x) = \int_0^x \boldsymbol{K}(x,t)\boldsymbol{y_0}(t)dt\\
&\boldsymbol{y_2}(x) = \int_0^x \boldsymbol{K}(x,t)(\boldsymbol{y_0}(t) + \boldsymbol{y_1}(t))dt - \int_0^x \boldsymbol{K}(x,t)\boldsymbol{y_0}(t)dt\\
&.\\
&.\\
&.\\
&\boldsymbol{y_m}(x) = \int_0^x \boldsymbol{K}(x,t)(\boldsymbol{y_0}(t) + \boldsymbol{y_1}(t) + \dots + \boldsymbol{y_m}(t))dt\\
&\qquad - \int_0^x \boldsymbol{K}(x,t)(\boldsymbol{y_0}(t) + \boldsymbol{y_1}(t) + \dots + \boldsymbol{y_{m-1}}(t))dt
\end{aligned}\right\} \tag{12}$$

where $\boldsymbol{f} = [f_1, f_2, \dots, f_n]^T$,

$\boldsymbol{y_0} = [y_{10}, y_{20}, \dots, y_{n0}]^T$,

$\boldsymbol{y_1} = [y_{11}, y_{21}, \dots, y_{n1}]^T$,

.

.

.

$\boldsymbol{y_m} = [y_{1m}, y_{2m}, \dots, y_{nm}]^T$, and

$$\boldsymbol{K} = \begin{pmatrix} K_{11} & \cdots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \cdots & K_{nn} \end{pmatrix}$$

Therefore, the solution is:

$y_1(x) = y_{10} + y_{11} + y_{12} + \cdots = \sum_{j=1}^{\infty} y_{1j}(x)$,

$y_2(x) = y_{20} + y_{21} + y_{22} + \cdots = \sum_{j=1}^{\infty} y_{2j}(x)$,

. .

.   .

.   .

$y_n(x) = y_{n0} + y_{n1} + y_{n2} + \cdots = \sum_{j=1}^{\infty} y_{nj}(x).$

When the method is applied to the nonlinear system (11), we have

$$\left.\begin{aligned}
\boldsymbol{y_0}(x) &= \boldsymbol{f}(x) \\
\boldsymbol{y_1}(x) &= \int_0^x \boldsymbol{K}(x,t,\boldsymbol{y_0}(t))dt \\
\boldsymbol{y_2}(x) &= \int_0^x \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t),\boldsymbol{y_1}(t)\big)dt - \int_0^x \boldsymbol{K}(x,t,\boldsymbol{y_0}(t))dt \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
\boldsymbol{y_m}(x) &= \int_0^x \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t),\boldsymbol{y_1}(t),\dots,\boldsymbol{y_m}(t)\big)dt \\
&\quad - \int_0^x \boldsymbol{K}(x,t,\boldsymbol{y_0}(t),\boldsymbol{y_1}(t),\dots,\boldsymbol{y_{m-1}}(t))dt
\end{aligned}\right\} \qquad (13)$$

where; $\boldsymbol{f} = [f_1, f_2, \dots, f_n]^T$,

$\boldsymbol{y_0} = [y_{10}, y_{20}, \dots, y_{n0}]^T$,

$\boldsymbol{y_1} = [y_{11}, y_{21}, \dots, y_{n1}]^T$,

.

.

.

$\boldsymbol{y_m} = [y_{1m}, y_{2m}, \dots, y_{nm}]^T$, and

$\boldsymbol{K} = [K_1, K_2, \dots, K_n]^T$,

The solution is given by;

$y_1(x) = \sum_{j=1}^{\infty} y_{1j}(x)$,

$y_2(x) = \sum_{j=1}^{\infty} y_{2j}(x)$,

.

.

.
$y_n(x) = \sum_{j=1}^{\infty} y_{nj}(x).$

4. Convergence of the New Iterative Method for System of Volterra Integral Equations.

Consider the system of nonlinear Volterra integral equation of the second kind in Eq.(11) above, where

$$\boldsymbol{f}(x) = [f_1(x),\ f_2(x), \dots, f_n(x)]^T,$$
$$\boldsymbol{y}(t) = [y_1(t),\ y_2(t), \dots, y_n(t)]^T, \text{ and}$$

$$\mathbf{K} = [K_1, K_2, \dots, K_n]$$

The kernels or nuclei $\mathbf{K} = [K_1, K_2, \dots, K_n]$ and the functions $f_i(x), i = 1,2,\dots,n$ are given real-valued functions. Now, suppose $|x-a| \le \beta$, $|t-a| \le \beta$, K is a continuous function of its arguments and satisfying a Lipschitz condition,

$$\|\boldsymbol{K}(x,t,\varphi) - \boldsymbol{K}(x,t,\phi)\| < L\|\varphi - \phi\|.$$

Let $\|\boldsymbol{K}(x,t,\varphi)\| \le M.$

Define

$$\boldsymbol{y_0}(x) = \boldsymbol{f}(x)$$

$$\boldsymbol{y_1}(x) = \int_a^x \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)\big)\,dt, \tag{14}$$

$$\boldsymbol{y_2}(x) = \int_a^x \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\boldsymbol{y_1}(t)\big)\,dt,$$

.

.

.

$$\boldsymbol{y_{m+1}}(x) = \int_a^x \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\boldsymbol{y_1}(t)+\cdots+\boldsymbol{y_m}(t)\big) - \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\boldsymbol{y_1}(t)+\cdots+\boldsymbol{y_{m-1}}(t)\big)dt \qquad m = 1,2,\dots$$

We prove $\sum_{i=1}^{\infty} y_i(x)$ is uniformly convergent.

$$\|\boldsymbol{y_1}(x)\| \le \int_a^x \big\|\boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)\big)\big\|dt \le M(x-a) \le M\alpha$$

$$\|\boldsymbol{y_2}(x)\| \le \int_a^x \big\|\boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\boldsymbol{y_1}(t)\big) - \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)\big)\big\|dt \le L\left\|\int_a^x \boldsymbol{y_1}(t)dt\right\|$$

$$\le ML\frac{(x-a)^2}{2!} \le \frac{M}{L}\frac{(K\alpha)^2}{2!},$$

$$\|\boldsymbol{y_3}(x)\| \le \int_a^x \big\|\boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\boldsymbol{y_1}(t)+\boldsymbol{y_2}(t)\big) - \boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\boldsymbol{y_1}(t)\big)\big\|dt \le L\left\|\int_a^x \boldsymbol{y_2}(t)dt\right\|$$

$$\leq ML^2 \frac{(x-a)^3}{3!} \leq \frac{M}{L}\frac{(K\alpha)^3}{3!}$$

.

.

.

$$\|\boldsymbol{y_{m+1}}(x)\| \leq \int_a^x \left\|\boldsymbol{K}\big(x,t,\boldsymbol{y_0}(t)+\cdots+\boldsymbol{y_m}(t)\big) - K\big(x,t,\boldsymbol{y_0}(t)+\cdots+\boldsymbol{y_{m-1}}(t)\big)\right\| dt$$

$$\leq L\left\|\int_a^x \boldsymbol{y_{m-1}}(t)dt\right\| \leq ML^m \frac{(x-a)^{m+1}}{(m+1)!} \leq \frac{M}{L}\frac{(K\alpha)^{m+1}}{(m+1)!}$$

Hence $\sum_{i=0}^{\infty} y_i(x)$ is absolutely and uniformly convergent and $y(x)$ satisfies (14).

5. Illustrative Examples

In order to illustrate the performance and accuracy of the new iterative method in the solution of linear and nonlinear systems of Volterra integral equations, we consider the following examples.

Example 5.1. Consider the system of linear Volterra integral equations of the second kind:

$$\left.\begin{aligned} y_1(x) &= x - x^2 + \cos x + \int_0^x \big(y_1(t)+y_2(t)\big)dt \\ y_2(x) &= x - \left(\tfrac{2}{3}\right)x^3 - \cos x + \int_0^x \big(ty_1(t)+ty_2(t)\big)dt \end{aligned}\right\} \quad (15)$$

The exact solutions are:

$$\big(y_1(x), y_2(x)\big) = (x+\cos x\,, x-\cos x)$$

Following the algorithm given in (12), the first few terms of $y(x)$ are:

$$y_{10}(x) = x - x^2 + \cos x,$$

$$y_{20}(x) = x - \left(\tfrac{2}{3}\right)x^3 + \cos x,$$

$$y_{11}(x) = \int_0^x \big(y_{10}(t)+y_{20}(t)\big)\,dt$$

$$= x^2 - \frac{1}{3}x^3 - \frac{1}{6}x^4$$

$$y_{21}(x) = \int_0^x \big(ty_{10}(t)+ty_{20}(t)\big)\,dt$$

$$= -\frac{1}{60}x^3(-40+15x+8x^2)$$

$$y_{12}(x) = \int_0^x \big((y_{10}+y_{11})+(y_{20}+y_{21})\big)\,dt - \int_0^x \big(y_{10}(t)+y_{20}(t)\big)\,dt$$

$$= -\frac{1}{45}x^6 - \frac{1}{12}x^5 + \frac{1}{12}x^4 + \frac{1}{3}x^3$$

$$y_{22}(x) = \int_0^x (t(y_{10} + y_{11}) + t(y_{20} + y_{21}))dt - \int_0^x \left(ty_{10}(t) + ty_{20}(t)\right) dt$$

$$= \frac{1}{15}x^5 - \frac{7}{72}x^6 - \frac{2}{105}x^7 + \frac{1}{4}x^4$$

$$y_{13}(x) = \frac{1}{15}x^5 - \frac{11}{840}x^7 - \frac{1}{360}x^6 - \frac{1}{420}x^8 - \frac{2}{105}x^7$$

$$y_{23}(x) = \frac{1}{18}x^6 - \frac{1}{420}x^7 - \frac{4}{960}x^8 - \frac{2}{945}x^9 + \frac{1}{15}x^7$$

$$y_{14}(x) = \frac{19}{2520}x^7 + \frac{1}{45}x^6 - \frac{13}{6720}x^8 - \frac{31}{20160}x^9 - \frac{1}{4725}x^{10} + \frac{1}{60}x^5$$

$$y_{24}(x) = \frac{2}{105}x^7 + \frac{19}{2880}x^8 - \frac{13}{7560}x^9 - \frac{31}{22400}x^{10} - \frac{2}{10359}x^{10}$$

.

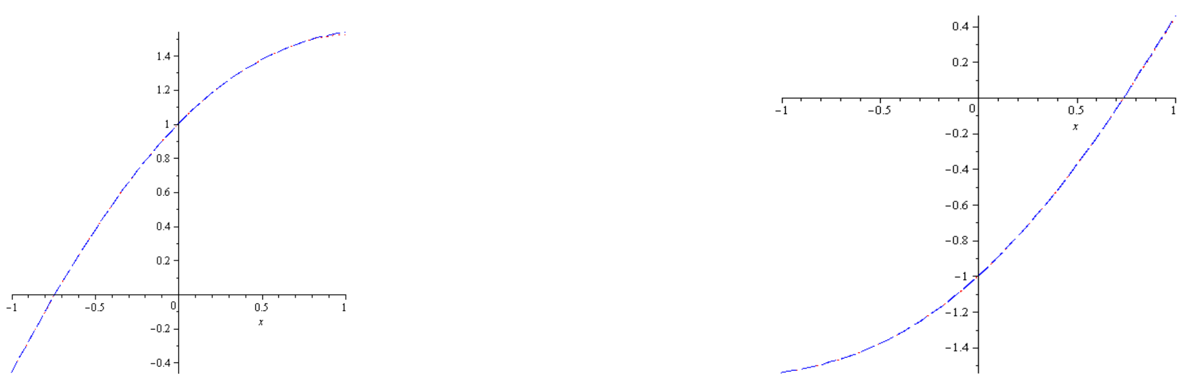
.

.

and so on, in the same manner, the rest components can be obtained. Thus, we get the solutions to be

$$y_1(x) = x + \cos x - \frac{1}{360}x^6 - \frac{1}{180}x^7 - \frac{29}{6720}x^8 - \frac{31}{20160}x^9 - \frac{1}{4725}x^{10} + \cdots$$

$$y_2(x) = x - \cos x - \frac{1}{420}x^7 - \frac{7}{1440}x^8 - \frac{29}{7560}x^9 - \frac{31}{22400}x^{10} - \frac{2}{10395}x^{11} + \cdots$$



(a) **(b)**

Figure 1: Exact and approximate solution for (a) $y_1(x)$ and (b) $y_2(x)$ in Equation (15), where red and blue represents the approximate and exact solution respectively.

Example 5.2. Consider the system of Volterra integral equations

$$\left.\begin{aligned} y_1(x) &= 1 - x^2 + \tfrac{1}{4}x^4 + \int_0^x \big(y_1(t) + y_2(t) - y_3(t)\big)\, dt, \\ y_2(x) &= 2x + x^2 - \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4 + \int_0^x \big(y_2(t) + y_3(t) - y_1(t)\big) dt, \\ y_3(x) &= -x - x^2 + x^3 - \tfrac{1}{4}x^4 + \int_0^x \big(y_3(t) + y_1(t) - y_2(t)\big) dt. \end{aligned}\right\} \quad (16)$$

whose exact solutions are $\big(y_1(x), y_2(x), y_3(x)\big) = (1 + x, x + x^2, x^2 + x^3)$.

Following the algorithm given in (12), the first few terms of $y(x)$ are:

$$y_{10}(x) = 1 - x^2 + \tfrac{1}{4}x^4,$$

$$y_{20}(x) = 2x + x^2 - \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4,$$

$$y_{30}(x) = -x - x^2 + x^3 - \tfrac{1}{4}x^4,$$

$$y_{11}(x) = \int_0^x \big(y_{10}(t) + y_{20}(t) - y_{30}(t)\big)\, dt$$

$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{20}x^5 + \tfrac{3}{2}x^2 - \tfrac{5}{12}x^4,$$

$$y_{21}(x) = \int_0^x \big(y_{20}(t) + y_{30}(t) - y_{10}(t)\big)\, dt$$

$$= \tfrac{1}{2}x^2 + x^3 + \tfrac{1}{12}x^4 - \tfrac{3}{20}x^5 - x,$$

$$y_{31}(x) = \int_0^x \big(y_{30}(t) + y_{10}(t) - y_{20}(t)\big)\, dt$$

$$= -\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{5}{12}x^4 + \tfrac{1}{20}x^5 + x,$$

$$y_{12}(x) = \int_0^x (y_{10} + y_{11} + y_{20} + y_{21} - y_{30} - y_{31})\, dt - \int_0^x \big(y_{10}(t) + y_{20}(t) - y_{30}(t)\big)\, dt$$

$$= -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{40}x^6 + \tfrac{7}{6}x^3 - \tfrac{3}{20}x^5,$$

$$y_{22}(x) = \int_0^x (y_{20} + y_{21} + y_{30} + y_{31} - y_{10} - y_{11})\, dt - \int_0^x \big(y_{20}(t) + y_{30}(t) - y_{10}(t)\big)\, dt$$

$$= -\tfrac{5}{6}x^3 + \tfrac{1}{4}x^4 + \tfrac{11}{40}x^5 - \tfrac{1}{40}x^6 - \tfrac{1}{2}x^2,$$

$$y_{32}(x) = \int_0^x (y_{30} + y_{31} + y_{10} + y_{11} - y_{20} - y_{21})\, dt - \int_0^x \big(y_{30}(t) + y_{10}(t) - y_{20}(t)\big)\, dt$$

$$= -\tfrac{1}{6}x^3 - \tfrac{5}{12}x^4 - \tfrac{1}{60}x^5 + \tfrac{1}{24}x^6 + \tfrac{3}{2}x^2,$$

$$y_{13}(x) = -\tfrac{5}{6}x^3 + \tfrac{11}{60}x^5 - \tfrac{11}{840}x^7 + \tfrac{1}{8}x^4 + \tfrac{1}{120}x^6,$$

$$y_{23}(x) = -\tfrac{13}{24}x^4 - \tfrac{1}{12}x^5 + \tfrac{19}{360}x^6 + \tfrac{1}{168}x^7 + \tfrac{1}{2}x^3,$$

$$y_{33}(x) = \tfrac{11}{24}x^4 - \tfrac{1}{12}x^5 - \tfrac{7}{120}x^6 + \tfrac{1}{168}x^7 + \tfrac{1}{2}x^3,$$

.
.
.

and so on, in the same manner, the rest components can be obtained. The 7-term approximate solution is given as:

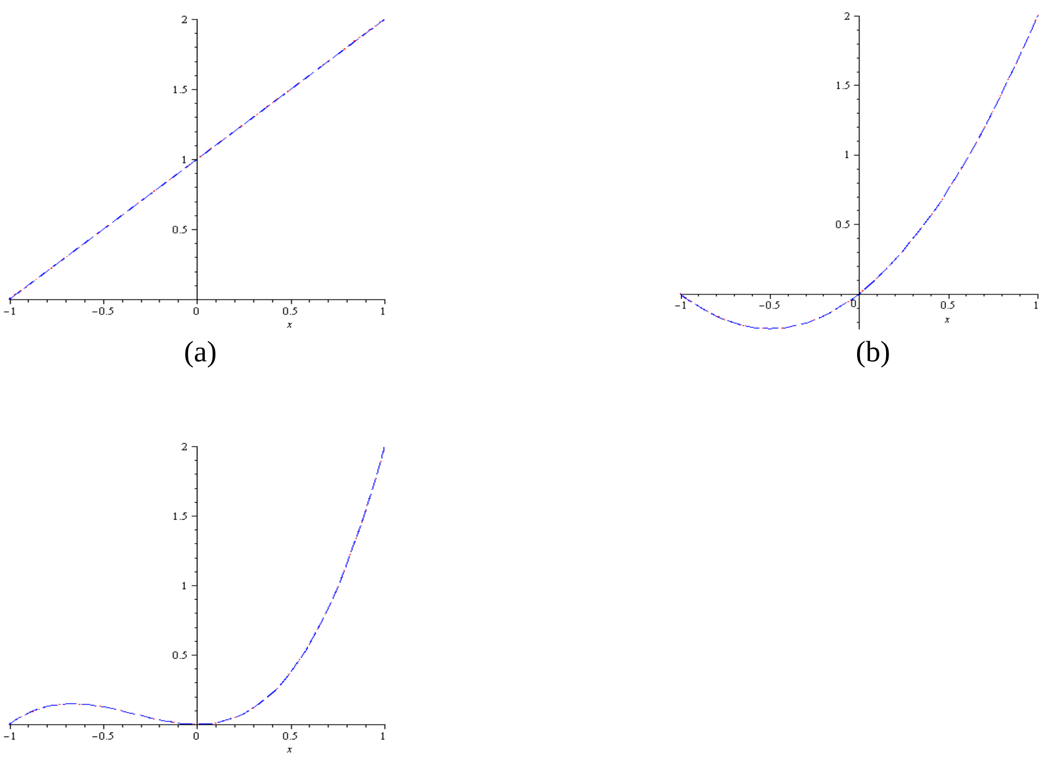
$$y_{16}(x) = \frac{42}{5040}x^7 - \frac{17}{36288}x^9 + \frac{17}{1330560}x^{11} + \frac{19}{4480}x^8 - \frac{341}{1814400}x^{10}$$

$$y_{26}(x) = \frac{43}{40320}x^8 + \frac{19}{20160}x^9 + \frac{43}{1814400}x^{10} - \frac{19}{739200}x^{11} - \frac{17}{1008}x^7$$

$$y_{36}(x) = -\frac{71}{13440}x^8 - \frac{17}{36288}x^9 + \frac{299}{1814400}x^{10} + \frac{17}{1330560}x^{11} + \frac{43}{5040}x^7$$

The solution in series form is given by

$$y_1(x) = 1 + x - \frac{43}{907200}x^{10} - \frac{43}{181440}x^9 + \frac{17}{1330560}x^{11} + \frac{1}{24}x^6 + \frac{17}{8064}x^8 + \cdots$$
$$y_2(x) = x + x^2 - \frac{43}{907200}x^{10} + \frac{17}{36288}x^9 - \frac{19}{739200}x^{11} + \frac{17}{8064}x^6 + \cdots$$
$$y_3(x) = x^2 + x^3 + \frac{17}{181440}x^{10} - \frac{43}{181440}x^9 + \frac{17}{1330560}x^{11} - \frac{19}{4480}x^6 + \cdots$$



(a) (b) (c)

Figure 2: Exact and approximate solution for (a) $y_1(x)$, (b) $y_2(x)$, (c) $y_3(x)$ of Eq.(15), where red and blue represents the approximate and exact solution respectively.

Example 5.3. Use the new iterative method to solve the following system of nonlinear Volterra integral equations.

$$y_1(x) = x - \frac{2}{3}x^2 + \int_0^x \left(y_1(t)^2 + y_2(t)\right) dt \quad (16)$$

$$y_2(x) = x^2 - \frac{1}{4}x^4 + \int_0^x \left(y_1(t) \times y_2(t)\right) dt$$

The new iterative method yeilds:

$$y_{10}(x) = x - \frac{2}{3}x^2$$

$$y_{20}(x) = x^2 - \frac{1}{4}x^4$$

$$y_{11}(x) = \int_0^x \left(y_{10}(t)^2 + y_{20}(t)\right) dt$$

$$= \frac{4}{63}x^7 - \frac{19}{60}x^5 - \frac{2}{3}x^3$$

$$y_{21}(x) = \int_0^x \left(y_{10}(t) * y_{20}(t)\right) dt$$

$$= \frac{1}{48}x^8 - \frac{11}{72}x^6 + \frac{1}{4}x^4$$

$$y_{12}(x) = \int_0^x \left(y_{10}(t)^2 + y_{11}(t)^2 + y_{20}(t) + y_{21}(t)\right) dt - \int_0^x \left(y_{10}(t)^2 + y_{20}(t)\right) dt$$

$$= \frac{16}{59535}x^{15} - \frac{38}{12285}x^{13} + \frac{361}{39600}x^{11} + \frac{149}{9072}x^9 - \frac{443}{2520}x^7 + \frac{19}{60}x^5$$

$$y_{21}(x) = \int_0^x \left(y_{10}(t) + y_{11}(t).y_{20}(t) + y_{21}(t)\right) dt - \int_0^x \left(y_{10}(t).y_{20}(t)\right) dt$$

$$= \frac{1}{12096}x^{16} - \frac{2957}{2540160}x^{14} + \frac{209}{51840}x^{12} + \frac{17}{2016}x^{10} - \frac{229}{2880}x^8 + \frac{11}{72}x^6$$

$$y_{13}(x) = \frac{256}{109876902975}x^{31} - \frac{1216}{21210236775}x^{29} + \frac{360278}{672354057375}x^{27} + \cdots$$

$$y_{23}(x) = \frac{1}{1440270720}x^{32} - \frac{279449}{14744771496000}x^{30} + \frac{134399141}{692021275545600}x^{28} + \cdots$$

$$y_{14}(x) = \frac{266019823}{1154829312000}x^{17} + \frac{71437037933}{584832213504000}x^{19} + \frac{233201079653}{3354184753920000}x^{21} + \cdots$$

$$y_{24}(x) = \frac{977606113}{8559323136000}x^{18} + \frac{9851926321}{175889387520000}x^{20} + \frac{520678178713}{1589367544934400000}x^{22} + \cdots$$

.

.

.

and so on, in the same manner, the rest of components can be obtained. The sum of the first five terms is:

$$y_1(x) = x + \frac{65536}{7605948298647865225 89375}x^{63} - \frac{622592}{142161422820410566243125}x^{61} + \cdots$$

$$y_2(x) = x^2 + \frac{1}{395631215397933484800 0}x^{64} - \frac{123275693}{9102992090762683143371250 0}x^{62} \ldots$$

It is clear that the iterations converge to the exact solutions $y_1(x) = x$ and $y_2(x) = x^2$ as the number of iterations become large.

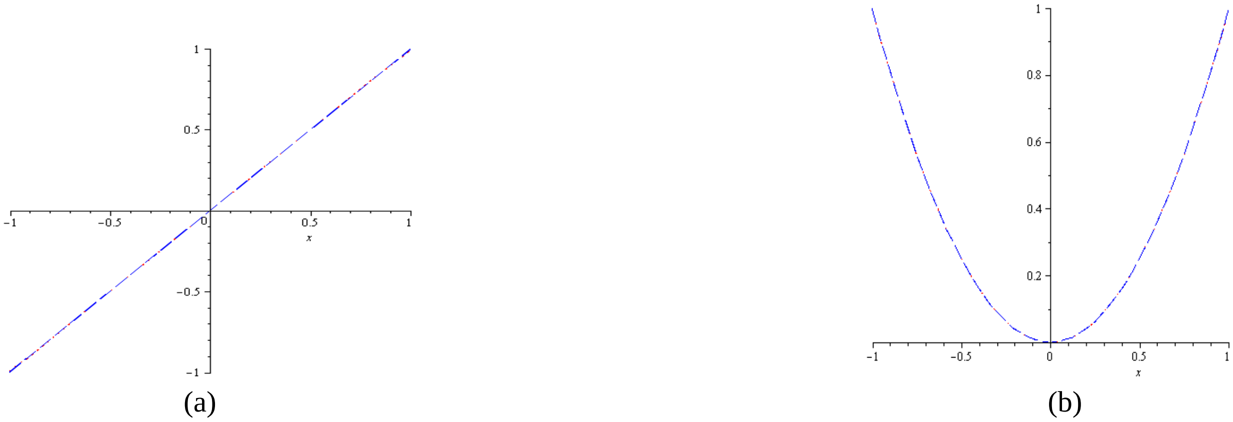


Figure 3: Exact and approximate solutions for (a) $y_1(x)$ and (b) $y_2(x)$ of Eq.(17), where the red and the blue represents the approximate and exact solutions respectively.

6. Conclusion

In this article, the new iterative method proposed by Daftardar-Gejji and Jafari (2006) has been successfully applied to find the approximate solutions of linear and nonlinear systems of Volterra integral equations. It is clear that the solutions agree well with the exact solutions for these equations.The results are plotted in Figures1 to 3, as well as the exact solutions. It can be concluded that, the new iterative method is a very powerful and efficient technique for finding approximate solutions for a wide class of problems. The method is simple in its principles and easy to implement in a computer.

The computations associated with the examples in this work were performed using Maple 13.

Table 1: Computed numerical results for Example 5.1.

x_i	$y_1(x_i)$approx	$y_1(x_i)$exact	$y_1(x_i)$error	$y_2(x_i)$approx	$y_2(x_i)$exact	$y_2(x_i)$error
-0.1	-0.459611715295881	-0.459697694131860	8.579×10^{-5}	-1.540138014141348	-1.540302305868140	1.643×10^{-4}
-0.9	-0.278416665991675	-0.278390031729336	2.663×10^{-5}	-1.521499747105309	-1.521609968270664	$1.1.2 \times 10^{-4}$
-0.8	-0.103346806864983	-0.103293290652835	5.351×10^{-5}	-1.496640161337669	-1.496706709347166	6.655×10^{-5}
-0.7	0.064797360292132	0.064842187284489	4.482×10^{-5}	-1.464806831593682	-1.464842187284488	3.536×10^{-5}
-0.6	0.225308267087964	0.225335614909678	2.735×10^{-5}	-1.425319623717159	-1.425335614909678	1.599×10^{-5}
-0.5	0.377569663773227	0.377582561890373	1.290×10^{-5}	-1.377576714816350	-1.377582561890373	5.847×10^{-6}
-0.4	0.521056466210800	0.521060994002885	4.528×10^{-6}	-1.321059410295026	-1.321060994002885	1.584×10^{-6}
-0.3	0.655335444530803	0.655336489125606	1.045×10^{-6}	-1.255336219676378	-1.255336489125606	2.694×10^{-7}
-0.2	0.780066461654490	0.780066577841242	1.1162×10^{-7}	-1.180066557983248	-1.180066577841242	1.986×10^{-8}
-0.1	0.895004163017142	0.895004165278026	2.260×10^{-9}	-1.095004165084842	-1.095004165278026	1.932×10^{-10}
0	1.000000000000000	1.000000000000000	0	-1.000000000000000	-1.000000000000000	0
0.1	1.095004161902955	1.095004165278026	3.375×10^{-9}	-0.895004165568708	-0.895004165278026	2.907×10^{-10}
0.2	1.180066317857665	1.180066577841242	2.600×10^{-7}	-0.780066622871552	-0.780066577841242	4.503×10^{-8}
0.3	1.255332953997767	1.255336489125606	3.535×10^{-6}	-0.655337412793754	-0.655336489125606	9.237×10^{-7}
0.4	1.321037455569531	1.321060994002885	2.354×10^{-5}	-0.521069239497157	-0.521060994002885	8.245×10^{-6}
0.5	1.377476851583247	1.377582561890373	1.057×10^{-4}	-0.377629089380443	-0.377582561890373	4.652×10^{-5}
0.6	1.424966234173678	1.425335614909678	3.694×10^{-4}	-0.225531638275912	-0.225335614909678	1.960×10^{-5}
0.7	1.463758209159493	1.464842187284488	1.084×10^{-4}	-0.065516194852746	-0.064842187284489	6.740×10^{-5}
0.8	1.493910251471526	1.496706709347166	2.796×10^{-4}	0.101298428023401	0.103293290652835	1.995×10^{-4}
0.9	1.515077452266361	1.521609968270664	6.532×10^{-4}	0.273129620577989	0.278390031729336	5.260×10^{-4}
1.0	1.526201776767611	1.540302305868140	1.141×10^{-3}	0.447043323039989	0.459697694131860	1.265×10^{-3}

Table 2: Computed numerical results for Example 5. 2

x_i	$y_1(x_i)$approx	$y_1(x_i)$exa	$y_1(x_i)$error	$y_2(x_i)$approx	$y_2(x_i)$exa	$y_2(x_i)$error	$y_3(x_i)$approx	$y_3(x_i)$exa	$y_3(x_i)$error
-0.1	0.000176817780984	0	1.768×10^{-3}	0.001617965367965	0	1.618×10^{-4}	-0.003923160173160	0	3.9×10^{-3}
-0.9	0.100071279631966	0.1000	7.128×10^{-4}	-0.089069608108149	-0.0900	9.304×10^{-4}	0.078866596643466	0.0810	2.1×10^{-3}
-0.8	0.200025621770467	0.2000	2.562×10^{-4}	-0.159513124126628	-0.1600	9.304×10^{-4}	0.126929000138354	0.1280	1.1×10^{-3}
-0.7	0.300007971992713	0.3000	7.972×10^{-5}	-0.209771715319004	-0.2100	4.869×10^{-4}	0.146512999721645	0.1470	4.8×10^{-4}
-0.6	0.400002055388675	0.4000	2.055×10^{-5}	-0.239906557349735	-0.2400	2.283×10^{-4}	0.143805037098390	0.1440	1.9×10^{-4}
-0.5	0.500000410350600	0.5000	4.104×10^{-6}	-0.249968009118145	-0.2500	9.344×10^{-5}	0.124934281396124	0.1250	6.6×10^{-5}
-0.4	0.600000056620288	0.6000	5.662×10^{-7}	-0.239991491779146	-0.2400	3.199×10^{-5}	0.095982699986440	0.0960	1.7×10^{-5}
-0.3	0.700000004362215	0.7000	4.362×10^{-8}	-0.209998472624976	-0.2100	8.508×10^{-6}	0.062996913454286	0.0630	3.1×10^{-6}
-0.2	0.800000000116225	0.8000	1.622×10^{-9}	-0.159999865323551	-0.1600	1.527×10^{-6}	0.031999728702102	0.0320	2.7×10^{-7}
-0.1	0.900000000000232	0.9000	2.32×10^{-11}	-0.089999997892338	-0.0900	1.347×10^{-7}	0.008999995759175	0.0090	4.2×10^{-8}
0	1.000000000000000	1.0000	0	0	0	0	0	0	0
0.1	1.099999999999759	1.1000	2.41×10^{-11}	0.110000002108598	0.1100	0	0.010999995758701	0.0110	4.2×10^{-9}
0.2	1.199999999874067	1.2000	1.26×10^{-10}	0.240000135155114	0.2400	2.109×10^{-8}	0.047999728459944	0.0480	2.7×10^{-7}
0.3	1.299999995078017	1.3000	4.922×10^{-9}	0.390001545725922	0.3900	1.352×10^{-6}	0.116996904170089	0.1170	3.1×10^{-6}
0.4	1.399999933439508	1.4000	6.656×10^{-8}	0.560008751680211	0.5600	1.546×10^{-5}	0.223982576805659	0.2240	1.7×10^{-5}
0.5	1.499999497074031	1.5000	5.029×10^{-7}	0.750033795759046	0.7500	8.752×10^{-5}	0.374933368119556	0.3750	6.7×10^{-5}
0.6	1.599997371409039	1.6000	2.627×10^{-6}	0.960102698433164	0.9600	3.376×10^{-5}	0.575800353118753	0.5760	2.0×10^{-4}
0.7	1.699989350221637	1.7000	1.065×10^{-5}	1.190265077463910	1.1900	1.027×10^{-4}	`0.832494377950569	0.8330	5.1×10^{-4}
0.8	1.799964199460038	1.8000	3.580×10^{-4}	1.440608215198403	1.4400	2.655×10^{-3}	1.150867577827925	1.1520	1.1×10^{-3}
0.9	1.899895666635837	1.9000	1.043×10^{-4}	1.711277253036667	1.7100	1.277×10^{-3}	1.536690983647338	1.5390	2.3×10^{-3}
1.0	1.999728385040885	2.0000	2.716×10^{-4}	2.002503507295174	2.0000	2.504×10^{-3}	1.995628407086740	2.0000	4.4×10^{-3}

Table 3: Computed numerical results for Example 5.3

x_i	$y_1(x_i)$approx	$y_1(x_i)$exact	$y_1(x_i)$error	$y_2(x_i)$approx	$y_2(x_i)$exact	$y_2(x_i)$error
-0.1	-1.000000000000000	-1.000000000000000	0	1.000000000000000	1.000000000000000	0
-0.9	-0.900000000000000	-0.900000000000000	0	0.810000000000000	0.810000000000000	0
-0.8	-0.800000000000000	-0.800000000000000	0	0.640000000000000	0.640000000000000	0
-0.7	-0.700000000000000	-0.700000000000000	0	0.490000000000000	0.490000000000000	0
-0.6	-0.600000000000000	-0.600000000000000	0	0.360000000000000	0.360000000000000	0
-0.5	-0.500000000000000	-0.500000000000000	0	0.250000000000000	0.250000000000000	0
-0.4	-0.400000000000000	-0.400000000000000	0	0.160000000000000	0.160000000000000	0
-0.3	-0.300000000000000	-0.300000000000000	0	0.090000000000000	0.090000000000000	0
-0.2	-0.200000000000000	-0.200000000000000	0	0.040000000000000	0.040000000000000	0
-0.1	-0.100000000000000	-0.100000000000000	0	0.010000000000000	0.010000000000000	0
0	0	0	0	0	0	0
0.1	0.100000000000000	0.100000000000000	0	0.010000000000000	0.010000000000000	0
0.2	0.200000000000000	0.200000000000000	0	0.040000000000000	0.040000000000000	0
0.3	0.300000000000000	0.300000000000000	0	0.090000000000000	0.090000000000000	0
0.4	0.400000000000000	0.400000000000000	0	0.160000000000000	0.160000000000000	0
0.5	0.500000000000000	0.500000000000000	0	0.250000000000000	0.250000000000000	0
0.6	0.600000000000000	0.600000000000000	0	0.360000000000000	0.360000000000000	0
0.7	0.700000000000000	0.700000000000000	0	0.490000000000000	0.490000000000000	0
0.8	0.800000000000000	0.800000000000000	0	0.640000000000000	0.640000000000000	0
0.9	0.900000000000000	0.900000000000000	0	0.810000000000000	0.810000000000000	0
1.0	1.000000000000000	1.000000000000000	0	1.000000000000000	1.000000000000000	0

7. References

Biazar, J. and Eslami, M.(2011). Differential Transform Method for Solving Systems of Volterra Integral Equations of the Second Kind with Comparison with Homotopy Perturbation Method. *International Journal of the Physical Sciences.* **6**(5): 1207-1212.

Biazar, J. and Ghazvini, H. (2009). He's Homotopy Perturbation Method for Solving Systems of Volterra Integral Equations of the Second Kind. *Choas Solitons and Fractals*. **39:** 770-777.

Daftardardar-Gejji, V. and Jafari, H. (2006). An Iterative Method for Solving Nonlinear Functional Equations. *Journal of Mathematical Analysis and Applications.* **316:** 753-763.

Hendi, A. (2011). Laplace Adomain Decomposition Method for Solving the Nonlinear Volterra Integral Equation with Weakly Kernels. *Studies in Nonlinear Sciences.* **2:** 129-134.

Hendi, F. A. and Al-Hazmi, S. (2010). The Nonlinear Volterra Integral Equation with Weakly Kernels and Toeplitz Matrix Method. *IJRRAS.* **3:** 216-232.

Jerri, A. M.(1999). Introduction to Integral Equations with Applications. New York: Wiely.

Lie-jun, X. (2013). A New Modification of Adomain Decomposition Method for Volterra Integral Equations of the Second Kind. *Hindawi Publishing Corporation, Journal of Applied Mathematics*. Article ID 795015.

Mohammad, R. Y. (2012). A numerical Method for Solving a System of Volterra Integral Equations. *World Applied Programming*. **2**(1)**:** 18-33.